Laws of motion in the central gravitational field

Vladimir Braun

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I. Kinematics. Generalization of Kepler's laws

The basic laws of motion of bodies in the central gravitational field are known as Kepler's laws of planetary motion. Kepler's laws are kinematic and were obtained empirically, based on the astronomical observations of Tycho Brahe. Here is the conclusion of the kinematic theory of motion in the central gravitational field, which is a generalization of Kepler's laws.

1. The universal equation of the trajectory

Using two forms of writing the derivative, by Newton and by Leibniz, let us write in the polar coordinate system (r, φ) the identities

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} \quad \text{and} \quad \dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}.$$

Excluding time from them,

$$\mathrm{d}\varphi = \dot{\varphi}\,\mathrm{d}t, \quad \mathrm{d}t = \frac{\mathrm{d}r}{\dot{r}} \quad \Rightarrow \quad \mathrm{d}\varphi = \frac{\dot{\varphi}}{\dot{r}}\mathrm{d}r,$$

and integrating the last equation, we obtain the integral

$$\varphi = \int \frac{\dot{\varphi}}{\dot{r}}\mathrm{d}r,$$

which is the universal equation of trajectory, one of two possible, in the polar coordinate system. Of course, for those cases where angular and radial velocities, $\dot{\varphi}$ and $\dot{r}$, are represented by functions of r and the integral can be taken.

2. Expressing velocity through radial and angular velocity

The vectors of radial and transversal velocity components are orthogonal, so the square of the velocity is the sum of the squares of these components:

$$v^2 = v_r^2 + v_\perp^2.$$

Since the radial component of velocity is simply the rate of change of coordinate r (radial velocity), and the transversal component of velocity is the rate of change of angular coordinate (angular velocity) multiplied by radius r,

$$v_r = \dot{r}, \quad v_\perp = r\dot{\varphi},$$

we have the following expression of velocity through radial and angular velocity:

$$v^2 = \dot{r}^2 + r^2\dot{\varphi}^2.$$

3. Representation of angular velocity as a function of distance

The law of conservation of momentum is equivalent to Kepler's second law, and means that the momentum of velocity is constant:

$$rv_{\perp} = r^2\dot{\varphi} = L\,.$$

From this we obtain the desired representation of angular velocity as a function of *r*:

$$\dot{\varphi} = \frac{L}{r^2}\,.$$

4. The equation of velocity as a function of distance

By replacing the angular velocity $\dot{\varphi}$ in the expression of velocity through its components,

$$v^2 = \dot{r}^2 + r^2\dot{\varphi}^2\,,$$

with its expression through momentum and distance, we obtain an equation that does not contain the angular coordinate φ:

$$v^2 = \dot{r}^2 + \frac{L^2}{r^2}\,.$$

The central gravitational field must admit stable bounded trajectories – planetary orbits. The motion of the planets occurs in a limited annular region, between the minimum and maximum distances from the center of gravity. The minimum and maximum distances of planetary orbits of the solar system are called the perihelion and aphelion of the orbit, as well as the pericenter and apocenter in the general case of an arbitrary central body, denote them as *p* and *a*.

When reaching the boundaries of the specified region, the radial component of velocity,

$$\dot{r} = \sqrt{v^2 - \frac{L^2}{r^2}}\,,$$

becomes zero. By equating the radial velocity to zero, we obtain an equation with respect to *r*, the roots of which are *a* and *p* – the apocenter and pericenter of the trajectory:

$$v^2r^2 - L^2 = 0\,.$$

Suppose the function $v^2r^2 - L^2$ is a polynomial of *r*. Having roots *a* and *p* means that the polynomial is divisible by $(a - r)$ and $(r - p)$, and has the representation:

$$v^2r^2 - L^2 = w(a - r)(r - p)\,,$$

where *w* is some polynomial. From this equality we obtain the equation for velocity:

$$v^2 = \frac{w(a-r)(r-p)}{r^2} + \frac{L^2}{r^2}\,,$$

or, after opening parentheses and combining terms by powers of *r*:

$$v^2 = -w + \frac{w(a+p)}{r} + \frac{L^2 - wap}{r^2}\,.$$

Taking the value of the squared velocity at an infinitely large distance from the center of gravity, assuming that the value of w in r_∞ remains finite, we obtain:

$$v_\infty^2 = -w .$$

That is, the polynomial -w is a constant – the square of the residual velocity.

5. Representation of radial velocity as a function of distance

Having the equation of velocity as a function of distance, we have at the same time the desired representation of radial velocity:

$$\dot{r} = \sqrt{v^2 - \frac{L^2}{r^2}} = \frac{\sqrt{w(a-r)(r-p)}}{r} .$$

6. Trajectory equation

Having obtained representations of angular and radial velocity as functions of distance,

$$\dot{\varphi} = \frac{L}{r^2} \quad \text{and} \quad \dot{r} = \frac{\sqrt{w(a-r)(r-p)}}{r} ,$$

we now substitute them into our universal trajectory equation:

$$\varphi = \int \frac{\dot{\varphi}}{\dot{r}} \mathrm{d}r = \int \frac{L}{r\sqrt{w(a-r)(r-p)}} \mathrm{d}r .$$

The integral is tabular. In the case of elliptic or hyperbolic velocity, when $wap > 0$, and when $(a+p)^2 > 4ap$, the solution is suitable: H. B. Dwight. Tables of integrals, No. 380.111, opt. 5. Writing down the solution in the arc cosine version, we obtain the equation of the trajectory as a function of $\varphi(r)$:

$$\varphi = \frac{L}{\sqrt{wap}} \arccos \frac{2ap - (a+p)r}{(a-p)r} + \varphi_0 .$$

To obtain the equation of trajectory as a function of $r(\varphi)$, we solve this equation with respect to r, and as a result, assuming $\varphi_0 = 0$, we obtain the following equation:

$$r = \frac{f}{1 + e\cos(i\varphi)} ,$$

where e is the eccentricity of the orbit, f is the focal parameter of the orbit, i is the pericenter shift parameter of the orbit, and

$$e = \frac{a-p}{a+p} ,\ f = \frac{2ap}{a+p} ,\ i = \frac{\sqrt{wap}}{L} .$$

The resulting equation is the equation of motion along a rotating conic section, ellipse or hyperbola. In the case of parabolic velocity, when $w = 0$ and $a = \infty$, the equation of velocity and the form of the trajectory remain undefined here.

Note that the equation can be written in four variants, with ±cos and with ±sin, which gives four options for the orientation of the trajectory, in particular the initial position of the pericenter of the trajectory: to the right, to the left, and above, below the center, respectively.

7. The integral of time as a function of distance

Let's start with the identity

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}.$$

From it we get:

$$\mathrm{d}t = \frac{1}{\dot{r}}\mathrm{d}r, \qquad t = \int \frac{1}{\dot{r}}\mathrm{d}r.$$

Substituting here the expression for the radial velocity as a function of the distance from the center of gravity, we obtain the following time integral:

$$t = \int \frac{r}{\sqrt{w(a-r)(r-p)}}\mathrm{d}r.$$

8. Circulation period

In general, the period of circulation is not the time of making a complete revolution equal to 2π, but the time during which a repeated section of the trajectory is passed, for example, the section from perihelion to aphelion and back to perihelion. Using the integral of time, we obtain a definite integral for the period of circulation

$$T = 2\int_p^a \frac{r}{\sqrt{w(a-r)(r-p)}}\mathrm{d}r.$$

The integral is tabular. For elliptic velocity, when $w > 0$, and when $(a + p)^2 > 4ap$, the solution is suitable: H. B. Dwight. Tables of integrals, No. 380.011, with reference to No. 380.001, variant 5. From where, writing down the solution in the arc cosine version, we obtain:

$$T = \frac{2}{\sqrt{w}}\left(\frac{a+p}{2}\arccos\frac{a+p-2r}{a-p} - \sqrt{(a-r)(r-p)}\right)\Bigg|_p^a,$$

that is

$$T = \frac{\pi(a+p)}{\sqrt{w}}.$$

9. Offset of the trajectory pericenter

The pericenter displacement is determined by the period of the function

$$r(\varphi) = \frac{f}{1 + e\cos(i\varphi)},$$

defining the trajectory. Its period, which coincides with the period of the function $\cos(i\varphi)$, in contrast to the period of the function $\cos(\varphi)$, is not equal to a complete revolution, 2π, but is equal to $2\pi/i$. The difference between the period of the function $r(\varphi)$ and the total revolution is the desired displacement:

$$\Delta\varphi = 2\pi\left(\frac{1}{i} - 1\right).$$

10. Gravitational acceleration

Having the equation of velocity as a function of distance,

$$v^2 = -w + \frac{w(a+p)}{r} + \frac{L^2 - wap}{r^2},$$

it is not difficult to obtain a similar equation of gravitational acceleration.
The acceleration is equal to the derivative over the distance of half the square of the velocity:

$$g = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v^2}{2}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(-w + \frac{w(a+p)}{r} + \frac{L^2 - wap}{r^2}\right),$$

that is,

$$g = -\left(\frac{w(a+p)}{2r^2} + \frac{L^2 - wap}{r^3}\right),$$

in agreement with Newton's theorem on the motion of bodies in moveable orbits [1].
The minus sign here indicates the direction of acceleration, and can be discarded if we are only interested in the absolute value of the acceleration.

II. Dynamics

All the laws obtained above are kinematic; they contain only distance, time, velocity, and acceleration, and do not contain mass, force, or energy. In order to obtain a dynamic physical theory from a kinematic theory, it is necessary to relate the dynamic parameters to the kinematic parameters in some way. There is no rule that prescribes exactly how this must be done, other than the requirement that the resulting theory correspond to reality.

1. Classical dynamics

In particular, if the kinematic expression of the gravitational acceleration corresponds to the expression of the gravitational acceleration of the classical gravity theory:

$$\frac{w(a+p)}{2r^2} + \frac{L^2 - wap}{r^3} = \frac{\mathrm{G}M}{r^2},$$

then, equating the coefficients of the same powers of r,

$$\frac{w(a+p)}{2} = \mathrm{G}M, \quad L^2 - wap = 0,$$

we obtain:

$$w = \frac{2\mathrm{G}M}{a+p}, \quad L = \sqrt{wap}.$$

Substituting these expressions into the equations of our kinematic theory, we obtain the laws of motion in the central gravitational field known from classical gravity theory:

$$v = \sqrt{2\mathrm{G}M\left(\frac{1}{r} - \frac{1}{a+p}\right)}, \; T = \pi\sqrt{\frac{(a+p)^3}{2\mathrm{G}M}}, \; r = \frac{f}{1 + e\cos(\varphi)}, \; \Delta\varphi = 0, \text{ et al.}$$

2. Modified dynamics

However, as we know, the classical theory of gravitation does not fully correspond to reality. The perihelion of the orbits of the planets, and in particular the perihelion of Mercury's orbit, shifts anomalously (i.e., in contradiction to theory).
For the dynamic theory to correspond to reality, it is necessary to relate dynamic parameters to kinematics in a different way, perhaps like this:

$$\frac{w(a+p)}{2r^2} + \frac{L^2 - wap}{r^3} = \frac{\mathrm{G}M}{r^2} + \frac{6\mathrm{G}^2M^2}{c^2r^3},$$

where c is the speed of light. In the modified dynamic theory of motion in the central field based on this relation, the displacement of the pericenter of the trajectory will be equal:

$$\Delta\varphi = 2\pi\left(\sqrt{1 + \frac{6\mathrm{G}M}{c^2 f}} - 1\right) \approx 2\pi\frac{3\mathrm{G}M}{c^2 f}.$$

The last formula has appeared more than once in the history of physics (as $\Delta\varphi = \dfrac{6\pi\mathrm{G}M}{c^2A(1-e^2)}$, where A is the semi-major axis of the orbit). It was first obtained by Paul Gerber in 1898. It is believed that this formula gives the displacement corresponding to the "observed" anomalous displacement of the perihelion of Mercury, and the other planets of the Solar System. Whether it is really so – we will probably know only when we learn the mechanism of gravitation, which still, as in Newton's time, remains for us a mystery.

III. Examples

1. Classical dynamics

In the tutorial by E. I. Butikov [2, p. 17] the problem is given:

> A ballistic projectile is launched vertically upward from the Earth's surface with an initial velocity whose modulus is equal to the circular velocity for the extremely low orbit (i.e., the first space velocity): $v_0 = v_1 = \sqrt{gR}$.
> How long does it take from launch to impact on the Earth?

The solution given there is indirect and overly complicated. With the help of our general formulas, the problem is solved directly, without any tricks.

Solution

1. A vertical trajectory of finite altitude is a degenerate elliptical orbit whose perigee coincides with the center of gravity, i.e., with the center of the Earth. Therefore, for the perigee of the orbit we have the value: $p = 0$.

2. By equating the orbital velocity at the Earth's surface, i.e. at distance R from the center, to the initial velocity, we obtain the equation

$$\sqrt{2\mathrm{G}M\left(\frac{1}{R} - \frac{1}{a}\right)} = \sqrt{\frac{\mathrm{G}M}{R}},$$

from which for the apogee we obtain the value: $a = 2R$.

3. Knowing the parameters of the orbit, the duration of the flight (ascent from $r = R$ to $r = 2R$ and fall back to Earth) can be calculated using the time integral:

$$t = 2\int_R^{2R} \frac{r}{\sqrt{w(a-r)(r-p)}}\,\mathrm{d}r\,.$$

Writing down the solution of the already familiar integral, we have:

$$t = 2\sqrt{\frac{a+p}{2\mathrm{G}M}}\left(\frac{a+p}{2}\arccos\frac{a+p-2r}{a-p} - \sqrt{(a-r)(r-p)}\right)\Bigg|_R^{2R},$$

from which we get the following result:

$$t = (\pi + 2)\sqrt{\frac{R^3}{\mathrm{G}M}}\,.$$

Comparing this expression with the period of circulation,

$$T = 2\pi\sqrt{\frac{R^3}{\mathrm{G}M}},$$

we obtain, in agreement with the original solution:

$$t = \frac{\pi+2}{2\pi}T = 0.82T\,.$$

2. Modified dynamics

> "OJ 287 is a double system of black holes, the largest of which has a mass equal to 18 billion solar masses, actually the mass of a small galaxy. A smaller companion weighs as much as 100 million solar masses. Its orbital period is 12 years".
>
> This object is interesting because of its huge shift of the pericenter of the orbit, which is about 39 degrees per one orbital period. Such a displacement, in contrast to the displacement of the perihelion of Mercury, can be shown graphically.

The masses of bodies of the system are comparable in magnitude, so here we have a two-body problem, which is reduced to the problem of motion in the central field. Let us solve it in a relative frame of reference connected with one of the bodies.

Solution

Let's make a system of equations:

$$\left\{T = \pi\sqrt{\frac{(a+p)^3}{2\mathrm{G}M}},\ \ \Delta\varphi = 2\pi\left(\sqrt{1+\frac{6GM}{c^2 f}} - 1\right)\right\},\ \text{where } M = m_1 + m_2 \text{ and } f = \frac{2ap}{a+p}\,.$$

Substituting the numerical data and solving the system with respect to a and p, we obtain the following parameters of the orbit of one companion with respect to the other:

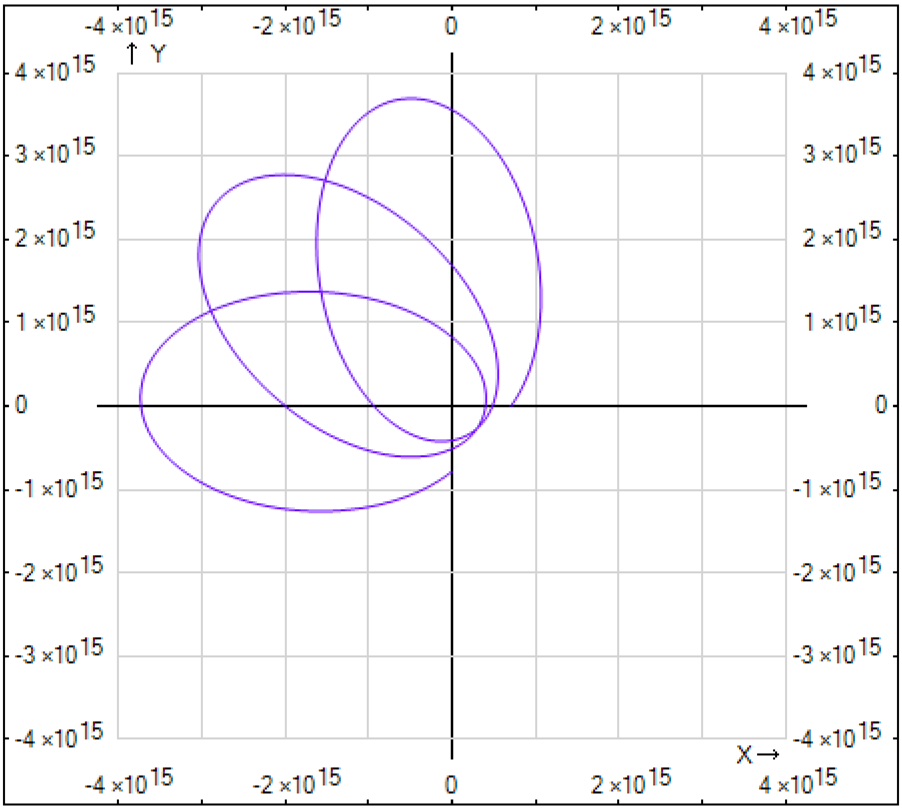
$a = 3.730\cdot 10^{15}$, $p = 3.876\cdot 10^{14}$,

$f = 7.022\cdot 10^{14}$, $e = 0.812$, $i = 0.902$.

By plotting the equation

$$r = \frac{7.022 \cdot 10^{14}}{1 - 0.812 \sin(0.902\,\varphi)}$$

in the interval [0; 5.5π] we obtain this trajectory:



References

1. Isaac Newton. [The Mathematical Principles of Natural Philosophy](). Of the motion of bodies in moveable orbits; and of the motion of the apsides. Proposition XLIV. Theorem XIV.
2. Бутиков Е. И. [Закономерности кеплеровых движений](). 2006.